Example 3.12: Two-Sheaf Topology on a DVR

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This is example 3.12 of Chapter II in [Mil80], p. 75, with the details filled in. A is a DVR, $X = \operatorname{Spec} A$. Let $\mathfrak{m}$ be the maximal ideal and η the zero ideal. We have the inclusion maps $i : \operatorname{Spec} k \to X$ and $j : \operatorname{Spec} K \to X$, where $k = A/\mathfrak{m}$ and $K = K(A) = A_{(0)}$. Sheaves on X are equivalent to triples $(F_1, F_2, \phi : F_1 \to i^* j_* F_2)$, where F_1 is a sheaf on k and F_2 a sheaf on K. Let B be the integral closure of A in K_s, a separable closure of K. Fix a prime ideal $\mathfrak{n}$ in B over $\mathfrak{m}$. This is equivalent to fixing an embedding of $\mathcal{O}_{X,\bar{\mathfrak{m}}} \cong B^I_{\mathfrak{n}^I} \hookrightarrow K_s$ [the fact that $\mathcal{O}_{X,\bar{\mathfrak{m}}} \cong B^I_{\mathfrak{n}^I}$ is in [Mil80, p. 38]], which is also equivalent to fixing a filtration $G_K \supset D \supset I \supset 1$, where $D = \{\sigma \in G_K \mid \sigma(\mathfrak{n}) = \mathfrak{n}\}$ is the decomposition group and $I = \{\sigma \in D \mid \sigma|_{B/\mathfrak{n}} = id\}$ is the inertia group, and we know $D/I \cong G_k$.

Theorem. *If the sheaf $F_2 = F_M$ corresponds to the G_K-module M, then $i^* j_* F_2$ corresponds to the G_k-module M^I.*

Proof. By [Tam94, p. 118] the G_k-module associated to $i^* j_* F_M$ is $(i^* j_* F_M)_{\bar{\mathfrak{m}}}$, so we just have to show this equals M^I. Following the suggestion given in [Mil80], we use theorem 3.2: $(i^* j_* F_M)_{\bar{\mathfrak{m}}} = (j^* F_M)_{\bar{\mathfrak{m}}} = \tilde{F}_M(\tilde{X}')$, where

$$\begin{aligned}
\tilde{X}' &= K \times_X \mathcal{O}_{X,\bar{\mathfrak{m}}} \\
&= K \times_X B^I_{\mathfrak{n}^I} \\
&= \operatorname{Spec}(K \otimes_A B^I_{\mathfrak{n}^I}) \\
&= \operatorname{Spec} K_s^I \ [\text{AM69}, 5.12] \\
&= \text{the maximal unramified extension of } K
\end{aligned}$$

and $\tilde{F}_M = f^* F_M$, where $f : \tilde{X}' \to K$ is the canonical map. Thus, the stalk is

$f^* F_M(\tilde{X}') = \varinjlim F_M(U)$ over K-maps $\tilde{X}' \to U, U$ étale over K

$$
\begin{aligned}
&= \varinjlim F_M(\operatorname{Spec} L) \text{ over finite extensions } L \text{ of } K \text{ contained in } K_s^I \\
&= \varinjlim \operatorname{Hom}_{G_K}(\operatorname{Hom}_{\operatorname{Spec} K}(\operatorname{Spec} K_s, \operatorname{Spec} L), M) \\
&= \varinjlim \operatorname{Hom}_{G_K}(\operatorname{Hom}_K(L, K_s), M) \\
&= \operatorname{Hom}_{G_K}(\varprojlim \operatorname{Hom}_K(L, K_s), M) \\
&= \operatorname{Hom}_{G_K}(\operatorname{Hom}_K(\varinjlim L, K_s), M) \\
&= \operatorname{Hom}_{G_K}(\operatorname{Hom}_K(K_s^I, K_s), M) \\
&= \operatorname{Hom}_{G_K}(G_K/I, M) \\
&= M^I
\end{aligned}
$$

as desired (note G_K/I is not a group since I is not normal in G_K, but it is still a G_K-set). □

Thus the category of sheaves on A is equivalent to the category of triples $(M, N, \phi : M \to N^I)$, where M is a G_k-module and N a G_K-module.

References

[AM69] M. F. Atiyah and I. G. MacDonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[Mil80] James S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.

[Tam94] Günter Tamme. *Introduction to étale cohomology*. Universitext. Springer-Verlag, Berlin, 1994. Translated from the German by Manfred Kolster.